HUMAN CAPITAL
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Philosophy and Methodology of Sciences

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Truth and proof

(lecture 4)

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Theoremhood

DEFINITION: THEOREM OF CPL

A wff α is a CPL theorem iff there is a finite sequence of wffs γ_1, ..., γ_k (proof), such that its last element is α ($\gamma_k = \alpha$) and each element is

either (1) a CPL axiom,

or (2) is derived from the proceeding elements by the rule MP.

Illustration

p → p CPL's law of identity

1. (p → (q → q)) → (p → p)	Ax. 1 α/ p → (q → q), β/p
2. p → (q → q)	Ax. 1 α/p, β/q
3. p → p	MP 1, 2

Provability (syntactic consequence) in CPL

DEFINITION: provability in CPL

A wff α is provable in CPL from the set Φ of wffs ($\Phi \vdash_p \alpha$) iff there is a finite sequence of wf's $\gamma_1, ..., \gamma_k$, such that its last element is α ($\gamma_k = \alpha$) and each element of the sequence is

either (1) an element of Φ (a premise),

or (2) an axiom of CPL,

or (3) is derived from preceding element by the rule MP

Provability (syntactic consequence) in CPL

$\{\alpha, \beta\} \vdash_p \alpha \wedge \beta$

1. α	premise
2. β	premise
$\alpha \quad \beta \quad \alpha \quad \gamma \quad \alpha \quad \beta \quad \gamma$	
3. $(\alpha \rightarrow \alpha) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow (\alpha \wedge \beta)))$	Ax. 7 β/α, γ/β

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$\alpha \quad \beta \quad \gamma \qquad \beta \quad \alpha \quad \gamma$

7. $(\alpha \rightarrow (\beta \rightarrow \beta)) \rightarrow (\beta \rightarrow (\alpha \rightarrow \beta))$	Ax. 4 γ/β

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Completeness of CPL

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COMPLETENESS THEOREM

For an arbitrary wff α of CPL:

α is a theorem of CPL iff α is a tautology of CPL.

Truth vs proof

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Tarski / Goedel result: for any sufficiently rich language, the set of true sentences is larger than the set of provable sentences (in any axiom system).

Truth vs proof

Truth vs. proof is an exam topic. Literature:

(1) Some logic textbook to explain what a formal language is and the machinery of Tarski's truth definition.

(2) Initial paragraphs of Tarski's classic "Truth in formalized languages" that explains the philosophy of his project (the Liar's paradox and some history of attempts to define truth)

(3) Tarski's paper "Truth and proof".