Philosophy and Methodology of Sciences

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Logic 3

(lecture 3)

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Reasoning and rules of reasoning

What reasonings are valid? We are interested in infallible reasonings, i.e., those that transmit truth: if its premises are true, its conclusion(s) are true as well.

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To characterize validity we need to extend the concept of valuations

Valuations in CPL

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$v(\alpha \equiv \beta) = 1$ iff $v(\alpha) = v(\beta)$

More on valuations

Definition: valuation of a set of formulas

A valuation v satisfies a set Φ of formulas iff v satisfies each formula of Φ.

Moral:

A valuation v does not satisfy a set Φ of formulas iff there is a formula α in Φ such that $v(\alpha) = 0$.

Fact: Every valuation satisfies the empty set.

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Fact: $\emptyset \vDash \alpha$ iff α is a tautology.

Normal rule

Definition: Normal rule

A rule with premises $\alpha_1, \ldots, \alpha_n$ and conclusion β is normal iff $\{\alpha_1, \ldots, \alpha_n\} \vDash \beta$.

Moral:

A rule with premises $\alpha_1, \ldots, \alpha_n$ and conclusion β is not normal iff there is a valuation v that satisfies the set $\{\alpha_1, \ldots, \alpha_n\}$ of premises but does not satisfy the conclusion β (i.e., $v(\alpha_1) = 1, \ldots, v(\alpha_n) = 1$, but $v(\beta) = 0$).

Normal rule

Checking if a reasoning is based on a normal rule:

Our servant is guilty or our gardener is guilty. Our driver is an accomplice provided that our servant is guilty. But if the gardener is guilty, then the driver is an accomplice. Thus, if the servant is guilty, then the gardner is guilty as well.

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Normal rule

How to check that the reasoning is based on a normal rule, i.e., if

$\{p \vee q, p \rightarrow r, q \rightarrow r\} \models (p \rightarrow q)$?

Directly, or indirectly by the tree-method. See the discussion group.

That finishes the truth-based approach to CPL

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- Towards proof-based approach: the idea is to introduce such a set of axioms that tautologies and only tautologies are provable form these axioms.
- Terminology: a theorem of the axiom system of CPL is an wff that has a proof in that system.
- A sought-for property: if a wff is tautological, it should be a theorem, and vice versa. In other words, the set of CPL tautologies is identical to the set of CPl theorems.This is known as completeness result.

Axiom schemata of CPL

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- Ax. 1. $\alpha \rightarrow (\beta \rightarrow \beta)$

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- Ax. 7. $(\alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow \gamma) \rightarrow (\alpha \rightarrow (\beta \wedge \gamma)))$

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- Ax. 7. $(\alpha \to \beta) \to ((\alpha \to \gamma) \to (\alpha \to (\beta \wedge \gamma)))$
- Ax. 8. $\alpha \to$

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- Ax. 5. $(\alpha \wedge \beta) \to \alpha$
- Ax. 6. $(\alpha \wedge \beta) \to \beta$
- Ax. 7. $(\alpha \to \beta) \to ((\alpha \to \gamma) \to (\alpha \to (\beta \wedge \gamma)))$
- Ax. 8. $\alpha \to (\alpha \vee \beta)$

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- Ax.10. $(\alpha \to \gamma) \to ((\beta \to \gamma) \to ((\alpha \vee \beta) \to \gamma))$
- Ax.11. $(\sim\alpha \to \sim\beta) \to (\beta \to \alpha)$

Axioms of CPL

Why schemata? Because actual axioms are derived by substitution: These are substitution for Ax. 1: $\alpha \rightarrow (\beta \rightarrow \beta)$

$\alpha \quad \beta \quad \beta$

$p \rightarrow (q \rightarrow q)$ $\alpha/p, \beta/q$

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$p \rightarrow (p \rightarrow p)$ $\alpha/p, \beta/p$

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$(p \wedge r) \rightarrow (q \rightarrow q)$ $\alpha/p \wedge r, \beta/q$

$\alpha \quad \beta \quad \beta$

$\sim p \rightarrow ((r \vee q) \rightarrow (r \vee q))$ $\alpha/\sim p, \beta/r \vee q$

RULES OF DERIVATION

primary rule: MP

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modus ponens

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$\alpha \rightarrow \beta$

RULES OF DERIVATION

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α

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$\alpha \rightarrow \beta$

α

β

Theoremhood

DEFINITION: THEOREM OF CPL

A wff α is a CPL theorem iff there is a finite sequence of wffs γ_1, ..., γ_k (proof), such that its last element is α ($\gamma_k = \alpha$) and each element is

either (1) a CPL axiom,

or (2) is derived from the proceeding elements by the rule MP.

Illustration

$p \to p$ CPL's law of identity

1. $(p \to (q \to q)) \to (p \to p)$ Ax. 1 α/ $p \to (q \to q)$, β/p
2. $p \to (q \to q)$ Ax. 1 α/p, β/q
3. $p \to p$ MP 1, 2